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Numerical solution of optimal control problems with constant control delays

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Abstract: We investigate a class of optimal control problems that exhibit constant exogenously given delays in the control in the equation of motion of the differential states. Therefore, we formulate an exemplary optimal control problem with one stock and one control variable and review some analytic properties of an optimal solution. However, analytical considerations are quite limited in case of delayed optimal control problems. In order to overcome these limits, we reformulate the problem and apply direct numerical methods to calculate approximate solutions that give a better understanding of this class of optimization problems.

In particular, we present two possibilities to reformulate the *delayed* optimal control problem into an *instantaneous* optimal control problem and show how these can be solved numerically with a state-of-the-art direct method by applying Bock's direct multiple shooting algorithm. We further demonstrate the strength of our approach by two economic examples.

Keywords: delayed differential equations, delayed optimal control, numerical optimization, time-to-build

JEL-Classification: C63, C61

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1 Introduction

Many intertemporal economic applications have the mathematical form of optimal control problems, where an objective function (e.g., intertemporal welfare, profit, costs, etc.) is sought to be maximized or minimized subject to a system of equations of motion, which determine the interaction of the stock and the control variables. Many economic systems do not react *instantly* but with a *delay* to changes in external influences (e.g., investment-lags, transportation-lags, etc.). Despite this ubiquitous experience, the analysis of such delayed dynamic systems in an optimal control framework has attracted little interest in economics so far.

The few exception are exclusively devoted to the delayed accumulation of capital, which means that investment needs time to turn into productive capital. El-Hodiri *et al.* (1972) derived a generalized maximum principle for a growth model with heterogeneous capital goods and exogenously given and constant delays of control and state variables. Production lags have been discussed in the macroeconomic real business cycle theory. Following an idea first given in Kalecki (1935), Kydland & Prescott (1982) empirically analyzed how far time consuming investment, which they called *time-to-build*, could explain business cycles observed in reality. Rustichini (1989) and Asea & Zak (1999) showed in simple optimal control models with one capital good (but a different lag structure) that the time-to-build feature is the driving force for the oscillatory system dynamics. More recently, Boucekkine *et al.* (2005) and Feichtinger *et al.* (2006) applied the time-to-build feature in the analysis of vintage capital models.

One reason for economists shunning delayed optimal control problems (especially in continuous time) is that they exhibit severe analytical and numerical difficulties. Even the linear approximation of the system dynamics around the stationary state is governed by a system of differential-difference equations of neutral type, which is, in general, *not* analytically solvable. Thus, the analytical discussion of delayed optimal control problems is limited to a few qualitative properties of the optimal solution (see, e.g., Rustichini 1989, Asea & Zak 1999 and Winkler 2004 for local stability analysis and qualitative systems dynamics). As a consequence, numerical optimization methods play an important role in analyzing and understanding the behavior of delayed optimal control problems.

In this paper we show how optimal control problems in continuous time with one stock and one control variable with a constant time delay can be solved numerically. We reformulate the original problem in two different ways into constrained control problems in ordinary differential equations with higher dimensional control functions respectively state variables. Thus, we avoid the solution of the delayed system at the cost of higher dimensionality. Furthermore, we show how to solve the reformulated control problems by *Bock's direct multiple shooting method.* The power of the solution method is demonstrated by treating two typical economic examples.

The remainder of the paper is structured as follows. Section 2 defines the class of delayed optimal control problems we seek to solve numerically. Furthermore, we review some qualitative properties of the optimal path and outline the difficulties for numerical solution methods. In section 3 we reformulate the optimal control problem in a suitable way to allow an application of the direct multiple shooting method. Two examples demonstrate the range of application for the solution method in section 4. Finally, section 5 concludes.

2 Problem formulation and analytical properties

We investigate a class of optimal control problems with one stock and one control variable and a control-delayed equation of motion of the stock variable. In this section we introduce a generic control problem and review some of its analytical properties.

2.1 A generic optimal control problem with delayed equation of motion

As usual in economic applications, we consider the maximization of an objective functional W, which is the discounted infinite integral over an autonomous felicity function F. With a stock variable x and a control variable u, the optimal control problem reads

$$\max_{x,u} W = \int_0^\infty F(x(t), u(t)) \exp[-\rho t] \, dt \tag{1a}$$

subject to

$$\dot{x}(t) = u(t-\sigma) - \gamma x(t) \ , \tag{1b}$$
$$u(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \ , \tag{1c}$$
$$x(0) = x_0 \ , \tag{1d}$$
$$u(t) = \xi(t) \ , \quad t \in [-\sigma, 0) \ , \tag{1e}$$

where ρ denotes the constant and positive discount rate, σ is a constant delay or time-lag, and γ is a constant decay rate. In addition, F is assumed to be twice continuously differentiable with respect to both arguments.

The crucial feature is that the control $u(\cdot)$ enters with a delay σ as $u(t-\sigma)$ in constraint (1b), while it is evaluated at time t as $u(t)$ in the objective functional (1a). In general, a differential equation with a delay in the state variables or control functions is referred to as a *delayed differential-difference equation* (DDE). Other common terms are *retarded linear functional differential equation* or *differential-difference equation of retarded type*. For an introduction to DDEs see Asea & Zak (1999: section 2) and Gandolfo (1996: chapter 27). A detailed exposition of (linear) functional differential equations is given in Bellman & Cooke (1963), Driver (1977), Hale (1977) and Kolmanovskii & Nosov (1986).

In contrast to models with instantaneous equations of motion, besides an initial value x_0 for the stock x, also an initial path ξ for the control $u(\cdot)$ in the time interval $[-\sigma, 0)$ has to be specified (or also optimized). Note that the path of the stock x in the time interval $t \in [0, \sigma]$ is completely determined by the initial stock x_0, the initial path $\xi(\cdot)$, and the retarded equation of motion in (1). Thus, optimal control problems which are governed by a retarded equation of motion exhibit an additional moment of inertia, as the variation of the stock reacts with a delay to the control.

Although the equation of motion is very specific, the maximization problem (1) represents numerous economic models. For example, it can be identified with capital accumulation models, where the capital stock accumulates delayed to investment, because investment needs some time to become productive capital. Other examples include pollution control problems, where the harmful pollution stock accumulates with a delay to the emissions of the respective pollutant, because the pollution does not accumulate at the same place where it is emitted and the transportation processes involved need time.

2.2 Necessary and sufficient conditions of the optimal solution

To derive necessary and sufficient conditions for the optimal solution of the optimization problem (1), we apply the generalized maximum principle derived in El-Hodiri *et al.* (1972) for delayed optimal control problems. This is possible, as the equation of motion (1b) is additively separable[1] in the time-lagged control variable $u(t-\sigma)$ and the stock variable $x(t)$. One obtains the *Hamiltonian* $\mathcal{H}$ as

$$\begin{aligned} \mathcal{H}(x(t), u(t)) &= F(x(t), u(t)) \exp[-\rho t] + p_x(t+\sigma)u(t) - \gamma p_x(t)x(t) \\ &\quad + p_\alpha(t)[u(t) - \alpha] + p_\beta(t)[\beta - u(t)] \ , \end{aligned} \tag{2}$$

where p_x denotes the costate variable or shadow price of the stock x, and p_α and p_β are the Kuhn-Tucker variables for the restrictions (1c) on the control variable u. Although it might look odd at first sight to have p_x evaluated at a future time, while we have a *retarded* equation of motion (1b), the explanation is quite intuitive: p_x measures the net present value of all future welfare gains/losses of one additional unit of the stock. As the reaction of the stock to the control takes the time period σ, a marginal increase in the control u at time t gives rise to a marginal unit of the stock x at time $t+\sigma$, of which the net present value is given by $p_x(t+\sigma)$.

Assuming that the Hamiltonian $\mathcal{H}$ is pointwise continuously differentiable with respect to the control u, the following necessary conditions hold for an optimal solution (partial derivatives are indicated by subscripts and only the time argument is stated explicitly):

$$\mathcal{H}_u = F_u(t) \exp[-\rho t] + p_x(t+\sigma) + p_\alpha(t) - p_\beta(t) = 0 \ , \tag{3}$$

$$\mathcal{H}_x = F_x(t) \exp[-\rho t] - \gamma p_x(t) = -\dot{p}_x(t) \ , \tag{4}$$

$$p_\alpha \geq 0 \ , \quad p_\alpha(t)[u(t) - \alpha] = 0 \ , \tag{5}$$

$$p_\beta \geq 0 \ , \quad p_\beta(t)[\beta - u(t)] = 0 \ . \tag{6}$$

These necessary conditions are also sufficient for a unique solution if the Hamiltonian $\mathcal{H}$ is strictly concave in both the stock x and the control u and, in addition, the following transversality condition is satisfied:

$$\lim_{t\to\infty} [p_x(t)x(t)] = 0 \ . \tag{7}$$

A sufficient condition for the strict concavity of the Hamiltonian $\mathcal{H}$ is that

$$F_{ii}(t) < 0 \quad \text{and} \quad \det[F_{ij}(t)] > 0 \ , \quad i, j = x, u \ , \tag{8}$$

[1] Recall that F is additively separable is equivalent to $F(x, u) = G(x) + H(u)$.

which we assume to hold in the following. The necessary condition (4) is an inhomogeneous linear first-order differential equation, which can be uniquely solved, together with the transversality condition (7), to yield:

$$p_x(t) = \int_t^\infty F_x(t') \exp[-\rho t'] \exp[-\gamma(t' - t)] \, dt' \ . \tag{9}$$

Hence, at the optimum the shadow price of the stock, p_x, equals the aggregated discounted future contributions to the objective function W of one additional marginal unit of the stock x. Condition (3) says that at the optimum, and as long as the restrictions on the control u are not binding (i.e., $p_\alpha = p_\beta = 0$), the marginal cost/benefit of one additional marginal unit of the control u equals the aggregated future benefit/cost of one additional marginal unit of the stock x. As one unit of u accumulates to the stock x delayed by the time-lag σ, the shadow price p_x has to be evaluated at time $t+\sigma$.

2.3 Optimal dynamic path and local stability analysis

Given that the restrictions (1c) on the control u are not binding (i.e., $p_\alpha = p_\beta = 0$), one obtains the following system of differential equations for an optimal solution from the necessary conditions (3) and (4), and the equation of motion for the stock x (1b):[2]

$$\begin{aligned} \dot{u}(t) &= \frac{F_u(t)}{F_{uu}(t)}(\gamma + \rho) + \frac{F_x(t+\sigma)}{F_{uu}(t)} \exp[-\rho\sigma] + \frac{F_{xu}(t)}{F_{xx}(t)}(\gamma x(t) - u(t-\sigma)) \ , \\ \dot{x}(t) &= u(t-\sigma) - \gamma x(t) \ . \end{aligned} \tag{10}$$

Note that $\dot{u}$ and $\dot{x}$ also depend on *advanced* (i.e., at a later time) and on *retarded* (i.e., at an earlier time) variables. Hence, (10) forms a system of *functional differential equations of neutral type*. Obviously, a possible approach to numerically solve the optimization problem (1) is to numerically solve the system of functional differential equations (10). However, recall that the system (10) is only the solution of the original optimization problem (1) in the case of an interior solution. Moreover, to determine a unique solution for (10) additional information about the first derivatives $\dot{x}$ and $\dot{u}$ at some point t is needed a priori. Therefore, we shall introduce a direct approach in this paper to numerically solve the original control problem (1) directly and, thus, does not depend on the exploitation of El-Hodiri *et al.*'s (1972) maximum principle.

Before we show how to reformulate the optimization problem (1) in order to derive a numerical solution, we state some of its analytical properties, which are derived in detail in Winkler (2004). First, the stationary state $(x^\star, u^\star)$ of a system of functional differential equations (10) is given by $\dot{x} = \dot{u} = 0$. This leads to the following (implicit) equations:

$$\begin{aligned} -\frac{F_x(x^\star, u^\star)}{F_u(x^\star, u^\star)} &= (\gamma + \rho) \exp[\rho\sigma] \ , \\ u^\star &= \gamma x^\star \ . \end{aligned} \tag{11}$$

[2] Differentiate (3) with respect to t, insert in (4) and solve for $\dot{u}$.

The stationary state $(x^\star, u^\star)$ exists and is unique if the felicity function F satisfies the Inada conditions (see Winkler 2004: Prop. 1).

Second, in order to determine the stability properties of the stationary state $(x^\star, u^\star)$, we examine the system dynamics in a neighborhood of the stationary state. Therefore, we linearize the system of functional differential equations (10) around the stationary state $(x^\star, u^\star)$ and analyze the characteristic equation of the resulting system of differential-difference equations. Denoting the characteristic roots by z and introducing the following abbreviations

$$A = \frac{F_{xu}(x^\star, u^\star)}{F_{uu}(x^\star, u^\star)} \exp[-\rho\sigma] \ , \quad B = \frac{F_{xu}(x^\star, u^\star)}{F_{uu}(x^\star, u^\star)} \ , \quad C = \frac{F_{xx}(x^\star, u^\star)}{F_{uu}(x^\star, u^\star)} \exp[-\rho\sigma] + \gamma(\gamma+\rho) \ ,$$

one obtains for the characteristic equation $Q(z) = 0$:

$$0 = z^2 - \rho z - A \exp[\sigma z](z + \gamma) + B \exp[-\sigma z](z - \rho - \gamma) - C \ . \tag{12}$$

The characteristic equation (12) is a *quasi-polynomial*, which has in general an infinite number of (complex) roots. However, the characteristic equation reduces to a simple quadratic equation with one positive and one negative real characteristic root for the special case that the partial derivative $F_{xu}(x^\star, u^\star) = 0$, and thus, $A = B = 0$.[3] A sufficient condition for $F_{xu}(x^\star, u^\star) = 0$ to hold is that the felicity function F is additively separable in the stock x and the control u.

In the general case, the characteristic roots are not analytically solvable. Nevertheless, the characteristic equation (12) can be shown to exhibit an infinite number of complex solutions with positive real parts and an infinite number of complex solutions with negative real parts (see Winkler 2004: Prop. 2). As a consequence, in either case the stationary state $(x^\star, u^\star)$ is a saddle point and, thus, for all initial stocks x_0 and all initial control paths ξ, there exists a unique optimal path which converges asymptotically towards the stationary state, unless the characteristic equation (12) exhibits purely imaginary roots (i.e., complex roots with vanishing real parts). In this case, the system dynamics may exhibit so called *limit-cycles*. That is, the optimal paths oscillate around the stationary state without converging towards or diverging from it. Limit-cycles in the case of delayed optimal control problems have been discussed by Rustichini (1989) and Asea & Zak (1999). If the roots are not purely imaginary, the optimal path shows exponentially damped convergence towards the stationary state. If the felicity function F is additively separable, we have monotonic convergence, otherwise oscillations may occur.

3 Numerical solution of the optimal control problem

Despite the analytical derivation of the qualitative properties of the optimal path, even the linearized approximation around the stationary state of the system of functional differential equations (10) is *not* analytically solvable. As a consequence, numerical optimization methods play an important role to analyze and understand the behavior of

[3] The characteristic equation also reduces to a quadratic equation in the trivial case $\sigma = 0$.

delayed optimal control problems. In the following section we show two ways how to reformulate the original problem in order to make it tractable for *Bock's direct multiple shooting method*, a highly efficient algorithm for the numerical solution of constrained optimal control problems in ordinary differential equations (ODE) and differential-algebraic equations (DAE).

3.1 Reformulation of the delayed optimal control problem

First, we have to restrict the time horizon for the numerical optimization to a finite value t_f, a caveat every numerical algorithm has to deal with. This poses no major problems as, according to the stability properties of the optimal solution outlined in the previous section, the results will be arbitrarily close to the problem with an infinite time horizon if t_f is sufficiently large. As we shall see, it is most convenient to set t_f to be a (large) multiple of the time-lag σ. In the delayed control problem (1) the delay σ solely appears in the control variable in the equation of motion (1b). Hence, it is possible to reformulate this delayed optimal control problem with one state variable into an instantaneous optimal control problem with several state variables. Thus, we can avoid to explicitly numerically treat the time-lag at the cost of higher dimensionality.

To see this, we split the time horizon t_f into n parts each the length of the delay σ and formulate the equation of motion separately in each of the resulting intervals. Thus, we obtain for the first interval $t \in [0, \sigma)$

$$\dot{x}(t) = \xi(t-\sigma) - \gamma x(t) \ , \qquad t \in [0, \sigma) \ , \tag{13}$$

where ξ is the initial control path in the time interval $t \in [-\sigma, 0)$. In the second interval $t \in [\sigma, 2\sigma)$ the equation of motion yields

$$\dot{x}(t) = u(t-\sigma) - \gamma x(t) \ , \qquad t \in [\sigma, 2\sigma) \ , \tag{14}$$

and so on.

The clue is to interpret each of the resulting DDEs as an independent differential equation. By introducing n new stock variables x_l and $n-1$ new control variables u_l with

$$x_l(t) = x(t+(l-1)\sigma) \ , \quad u_l(t) = u(t+(l-1)\sigma) \ , \quad t \in [0, \sigma) \ , \tag{15}$$

we achieve the following system of ordinary differential equations:

$$\begin{aligned}
\dot{x}_1(t) &= \xi(t-\sigma) - \gamma x_1(t) \ , \qquad t \in [0, \sigma) \ , \\
\dot{x}_2(t) &= u_1(t) - \gamma x_2(t) \ , \qquad t \in [0, \sigma) \ , \\
&\ \vdots \\
\dot{x}_{n-1}(t) &= u_{n-2}(t) - \gamma x_{n-1}(t) \ , \qquad t \in [0, \sigma) \ , \\
\dot{x}_n(t) &= u_{n-1}(t) - \gamma x_n(t) \ , \qquad t \in [0, \sigma) \ ,
\end{aligned} \tag{16}$$

Thus, we can reformulate the original optimization problem (1) as:

$$\max_{u_l, x_l} \int_0^\sigma \sum_{l=1}^{n} F(x_l(t), u_l(t)) \exp[-\rho(t + \sigma(l-1))] \, dt \tag{17a}$$

subject to

$$
\begin{aligned}
\dot{x}_1(t) &= \xi(t-\sigma) - \gamma x_1(t) \ , \\
&\vdots \\
\dot{x}_n(t) &= u_{n-1}(t) - \gamma x_n(t) \ ,
\end{aligned} \tag{17b}
$$

and the restrictions for the control variables u_l:

$$
u_l(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \ . \tag{17c}
$$

Furthermore we have to introduce additional constraints for the stock variables x_l at time $t = 0$ and $t = \sigma$ to ensure the continuity of the stock variable x of the original problem:

$$
x_l(\sigma) = x_{l+1}(0) \ , \quad l = 1, \ldots, n-1 \ . \tag{17d}
$$

Finally, the condition (1d) for the initial stock x_0 translates into

$$
x_1(0) = x_0 \ . \tag{17e}
$$

Note that we need only to determine $n-1$ control paths in the interval $[0, \sigma]$ as the optimal path for the stock in the interval $t \in [(n-1)\sigma, n\sigma)$ is completely determined by the stock at $t = (n-1)\sigma$, $x_{n-1}(\sigma)$, the control $u_{n-1}(t)$ and the equation of motion.

Remark 1. In addition to transforming the retarded optimization problem in a suitable form for numerical solution methods, the reformulation (17) also gives an intuitive explanation why the optimal control problem (1)

(i) exhibits an infinite number of characteristic roots in general, and

(ii) exhibits only two characteristic roots in the case that the felicity function F is additively separable.

To see (i), recall that the characteristic equation for an optimal control problem with n stock variables is a polynomial of order $2n$, which has in general $2n$ characteristic roots (although it may be less than $2n$ distinct roots as there may be multiple roots). Independent of the time-lag σ, n tends to infinity if we extend the time horizon $t_f \to \infty$. Thus, for an infinite time horizon t_f, the retarded optimization problem (1) with one stock variable is equivalent to an ordinary optimal control with an infinite number of stock variables, resulting in a characteristic equation with an infinite number of characteristic roots.

To see (ii), recall that F is additively separable is equivalent to $F(x, u) = G(x) + H(u)$. Thus, the objective functional (17a) yields for an infinite time horizon

$$
\max_{u_l, x_l} \int_0^\sigma \sum_{l=1}^{\infty} [G(x_l(t)) + H(u_l(t))] \exp[-\rho(t + \sigma(l-1))] \, dt \ . \tag{18}
$$

$G(x_1(t))$ is independent of variations in the control variables u_l, $l \geq 1$, as it is completely determined by the initial path ξ, the initial stock x_0 and the equation of motion. Therefore, it is sufficient to maximize the objective functional without the term exhibiting $G(x_1(t))$. Hence, we can rearrange the remaining terms to yield:

$$\max_{u_l, x_l} \int_0^{\sigma} \sum_{l=2}^{\infty} [G(x_l(t)) + H(u_{l-1}(t)) \exp[\rho\sigma]] \exp[-\rho(t + \sigma(l-1))] \, dt \tag{19}$$

Transforming the objective function back to one stock and one control variable yields:

$$\max_{u,x} \int_0^{\infty} [G(x(t+\sigma)) \exp[-\rho\sigma] + H(u(t))] \exp[-\rho t] \, dt \tag{20}$$

Introducing a new stock variable $\hat{x}(t) = x(t+\sigma)$ we achieve the following ordinary optimal control problem:

$$\max_{u,\hat{x}} \int_0^{\infty} [G(\hat{x}(t)) \exp[-\rho\sigma] + H(u(t))] \exp[-\rho t] \, dt \tag{21a}$$

subject to

$$\dot{\hat{x}}(t) = u(t) - \gamma \hat{x}(t) \; , \tag{21b}$$

$$u(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \; , \tag{21c}$$

$$\hat{x}(0) = x_\sigma \; , \tag{21d}$$

where x_σ is the value of the original stock variable x at time σ (which is completely determined by x_0, ξ and the original equation of motion). Thus, the retarded optimal control problem (1) is formally equivalent to the ordinary optimal control problem (21) with one stock and one control variable. As a consequence, its characteristic equation is a polynomial of second order, which is known to exhibit two characteristic roots.

Remark 2. Despite the intuitive explanation for the qualitative system dynamics in the general case and in the case of an additively separable felicity function F, the reformulation (17) does not promote the analytical derivation of the optimal solution in the general case. This holds as the additional coupled boundary constraints (17d), which guarantee the continuity of the original stock variable x, pose severe obstacles for an analytical solution.

Problem (17) is useful for analytical considerations as outline d in Remark 1 and can be solved by the direct multiple shooting method as will be shown in section 3.2. However, for a given time horizon t_f, the number n of differential state and control functions becomes quite large for small values of the time-lag σ. Therefore, we also consider another reformulation of the problem (1) with fixed dimension of state and controls.

To this end we introduce an additional control function. While $u_2(t)$ is the same as $u(t)$ before and denotes the control at time t, $u_1(t)$ represents the retarded control $u(t-\sigma)$.

Thus, u_1 and u_2 are coupled by $u_1(t) = u_2(t-\sigma)$ for $t \geq \sigma$ and $u_1(t) = \xi(t)$ for $0 \leq t \leq \sigma$. Then, problem (1) is equivalent to

$$\max_{u_1,u_2,x} \int_0^\infty F(x(t), u_2(t)) \exp[-\rho(t)]\, dt \tag{22a}$$

subject to

$$\dot{x}(t) = u_1(t) - \gamma x(t) \tag{22b}$$
$$u_1(t), u_2(t) \in [\alpha, \beta], \quad \alpha, \beta \in \mathbb{R} \tag{22c}$$
$$x(0) = x_0\ , \tag{22d}$$
$$u_1(t) = \xi(t-\sigma), \quad 0 \leq t < \sigma, \tag{22e}$$
$$u_1(t) = u_2(t-\sigma), \quad t \geq \sigma. \tag{22f}$$

Problem (22) still contains a retarded term, but it has moved from the differential equation (22b) to a constraint on the controls (22f), that can be dealt with efficiently by the direct multiple shooting method. In contrast to the reformulation (17), only one additional control variable has been introduced independently of the time horizon t_f and the time-lag σ.

3.2 Bock's direct multiple shooting method

In order to solve the reformulated optimal control problems (17) and (22) numerically, we apply the *direct multiple shooting* method originally developed by Bock and his coworker Plitt (1981), Bock & Plitt (1984). Let us consider an optimal control problem of the form

$$\max_{u,x} \int_{t_0}^{t_f} L(x(t), u(t))\, dt \tag{23a}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [t_0, t_f], \tag{23b}$$
$$0 \leq c(x(t), u(t)), \quad t \in [t_0, t_f], \tag{23c}$$
$$0 = r^{\text{eq}}(x(\tau_0), x(\tau_1), \ldots, x(\tau_m)), \tag{23d}$$
$$0 \leq r^{\text{ieq}}(x(\tau_0), x(\tau_1), \ldots, x(\tau_m)), \tag{23e}$$

with all occurring functions twice differentiable.

We approximate the n_u-dimensional control function $u(\cdot)$ by functions with local support and finitely many parameters. To this end we introduce a time grid

$$t_0 = \tau_0 < \tau_1 < \cdots < \tau_m = t_f \tag{24}$$

and split the time horizon $[t_0, t_f]$ into m so called *multiple shooting intervals* $[\tau_{j-1}, \tau_j]$, where $j = 1, \ldots, m$. On each multiple shooting interval we define a linear approximation $\phi^j(t)$ of the controls $u(t)$ by

$$\phi^j(t) := q_1^j + q_2^j t\ , \quad t \in [\tau_{j-1}, \tau_j]\ , \tag{25}$$

with vector valued parameters q^j.

We introduce m variables $s^j \in \mathbb{R}^{n_x}$ as initial values for the differential states on each multiple shooting interval $[\tau_{j-1}, \tau_j]$. The ODE (23b) is solved independently on every interval with initial values

$$x(\tau_j) = s^j \ , \quad j = 0, \ldots, m-1 \ . \tag{26}$$

To ensure continuous state trajectories $x(\cdot)$, the values at the end of interval j, obtained by integration with fixed initial value s^j, have to coincide with the initial state vector of the next interval $j+1$:

$$x(\tau_{j+1}; s^j) = s^{j+1} \ , \quad j = 0, \ldots, m-1 \ . \tag{27}$$

The so called *matching conditions* (27) allow to eliminate the additional degrees of freedom introduced by the supplementary optimization parameters s^j by *condensing* (for details see Bock & Plitt (1984)). Note that the conditions (27) are required to be satisfied only at the final solution of the problem and not during intermediate iterations of the optimization algorithm. Therefore, the direct multiple shooting method is also referred to as an *all-at-once-approach*, solving the simulation and optimization task at the same time. This allows to incorporate expert knowledge about the trajectory behavior into the initial values of the state trajectory and typically leads to good convergence properties of the method. The path and control constraints (23c) have to hold on the whole time interval $[t_0, t_f]$. To deal with this numerically, the direct multiple shooting method relaxes it to a constraint that is only evaluated on a finite time grid.

Following these lines, problem (23) is now an optimization problem in the variables q^j and s^j. It contains equality constraints that stem from the interior point equality constraints (23d) and the matching conditions (27), and inequality constraints that stem from the interior point equality constraints (23e) and the discretized path constraints (23c).

Subsuming all variables s^j and q^j into $w \in \mathbb{R}^{n_w}$ and rewriting the objective function as well as the constraints in adequate functions F, G and H, we obtain a non-linear program (NLP)

$$\min_w F(w) \quad \text{subject to} \quad \left\{ \begin{array}{rcl} G(w) & = & 0 \\ H(w) & \geq & 0 \end{array} \right. , \tag{28}$$

that can be solved by tailored methods. For example, by sequential quadratic programming (SQP) in combination with an efficient evaluation of all occurring functions, and the generation of derivatives, for example, by internal numerical differentiation. See Leineweber *et al.* (2003) for details and further references.

Now, let us consider an application of the direct multiple shooting method to the reformulations (17) and (22) of the original problem (1). Obviously, (17) is of the form (23) and can, thus, be solved with the direct multiple shooting method as described above. However, reformulation (22) contains an additional constraint (22f), which is not contained in the standard problem formulation (23).

Here, the approximation of the control functions allows to guarantee (22f) – if the corresponding entries of $u_1(t)$ in q^j and the ones of $u_2(t)$ in q^{j-1} match at all times τ_j, then the equation holds on the whole time horizon (as each piecewise linear control is uniquely determined by two points). If we extend the interior point equality constraint (23d) to allow also for arguments $u(\tau_j)$ (which is typically omitted, as only measurable influence of a control function shall be considered), then the direct multiple shooting method can be applied to solve both problems (17) and (22).

4 Examples

In the following we illustrate the potential of the numerical solution method described in the previous section by two examples, which stem from our research on delayed optimal control problems. The first example shows how numerical optimization can be used to analyze the transition from instantaneous to delayed stock accumulation. The second example focuses on the influence of the initial path ξ on the optimal paths of a delayed optimal control problem.

4.1 The transition from instantaneous to delayed capital accumulation

The first example is an optimal control capital accumulation model with an exogenously given delay between investment and capital accumulation, which is discussed in detail in Winkler *et al.* (2005).

Consider an economy with one non-producible input factor, for example, labor, which is given in constant amount $\bar{l}$ and distributed to three linear-limitational production processes. The first process produces one unit of the consumption good with one unit of labor. The second process combines λ units of labor together with κ units of capital to produce one unit of the consumption good. The third process creates one unit of investment from one unit of labor. Thus, we derive

$$\begin{aligned} c_1(t) &= l_1(t) \ , && (29)\\ c_2(t) &= \min\left[\frac{l_2(t)}{\lambda}, \frac{k(t)}{\kappa}\right] \ , && (30)\\ i(t) &= l_3(t) \ , && (31) \end{aligned}$$

where l_i denote the amount of labor employed in process i ($i = 1, 2, 3$). Assuming efficient production (i.e., $l_2(t)/\lambda = k(t)/\kappa$), and that the labor restriction holds with equality (i.e., $\sum_i l_i(t) = \bar{l}\ \forall\, t$), total consumption $c(t) = c_1(t) + c_2(t)$ yields:

$$c(t) = \bar{l} + \frac{1-\lambda}{\kappa}k(t) - i(t) \ . \qquad (32)$$

Further, we assume that investment at time t increases the capital stock k delayed at time $t+\sigma$, and that the capital stock deteriorates at the positive and constant rate γ

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t) \ . \qquad (33)$$

In addition, we assume that the capital stock k cannot be consumed (i.e., $i(t) \geq 0$). Assuming that the objective is to maximize intertemporal welfare, which is the discounted infinite integral of instantaneous welfare $V(c(t))$, the optimal control problem reads:

$$\max_{i(t)} \int_0^\infty V\left(\bar{l} + \frac{1-\lambda}{\kappa} k(t) - i(t)\right) \exp[-\rho t]\, dt \tag{34a}$$

subject to

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t)\ , \tag{34b}$$

$$i(t) \geq 0\ , \tag{34c}$$

$$\bar{l} - \frac{\lambda}{\kappa} k(t) - i(t) = c(t) - \frac{1}{\kappa} k(t) \geq 0\ , \tag{34d}$$

$$i(t) = \xi(t) = 0\ , \quad t \in [-\sigma, 0)\ , \tag{34e}$$

$$k(0) = 0\ . \tag{34f}$$

The restriction (34d) ensures that $c_1 \geq 0$. When it is binding, all labor is used to employ and maintain the capital stock. This implies that the consumption good is exclusively produced by the capital intensive process (30). For the following calculations we choose $V(c(t)) = \ln c(t)$, $\bar{l} = 26\frac{2}{3}$, $\lambda = 0.8$, $\kappa = 0.3$, $\gamma = 0.15$, $\rho = 0.1$, $t_f = 60$, $k_0 = 0$ and the initial path $\xi(\cdot) \equiv 0$.

The resulting optimization problem (34) is almost equivalent to the problem (1) discussed in section 2. As the additional inequality constraint (34d) fits directly into the definition of path and control constraints (23c), both reformulations (17) and (22) of (34) can be solved by the direct multiple shooting method.

Whereas the optimal solutions of the two different reformulations are, of course, identical, they exhibit different computational performance. Table 1 shows a comparison between the two approaches. All computations have been performed with the state-of-the-art optimal control software package `MUSCOD-II`, see Leineweber (1999), on a Pentium notebook with 1.5 GHz. Note that for the calculations the underlying control discretization grid has been chosen identical to the equidistant grid with distance σ. The computation times are given in seconds and describe how long it took before an accuracy of 10^{-6} of the Karush-Kuhn-Tucker (KKT) conditions was achieved. Obviously, problem reformulation (22) is much more suited for small time lags σ. The number of variables n_w of the non-linear program (NLP) is not the crucial indicator, though, as can be seen in table 1. Let us investigate in more detail what happens. Table 2 shows the distribution of the computing times for specific tasks. The times spent on condensing, online graphics, constraint reductions and other calculations are more or less the same. Also the time spent on state integration is compared to the rest.

The main difference is in the required time for calculating derivative information by internal numerical differentiation and the solution of the condensed quadratic programs (QPs). The size of the Jacobian matrix needed to calculate the sensitivities depends on the number of variables and is, thus, much higher for (17) than for (22). This effect can be reduced by a factor of about four by exploiting sparsity[4] (compare middle column in

[4] A matrix is called *sparse* if it contains only few nonzero entries, otherwise it is called *dense*.

Delay σ	(17) dense			(17) sparse			(22)		
	n_w	iters	time	n_w	iters	time	n_w	iters	time
0.5	605	47	208	605	47	110	724	20	10
0.4	755	50	419	755	50	224	904	23	24
0.3	1005	50	1094	1005	50	521	1204	23	53
0.2	1505	—	—	1505	—	—	1804	23	287
0.1	3005	—	—	3005	—	—	3604	14	1331

Table 1: Comparison of the number of variables n_w of the resulting NLP, number of SQP iterations and computing time in seconds needed to reach a KKT tolerance of 10^{-6}.

Action	(17) dense		(17) sparse		(22)	
	time	percent	time	percent	time	percent
Sensitivity generation	122	60.4%	30.0	26.7%	2.2	9.9%
State integration	0.7	0.3%	0.5	0.4%	0.8	4.1%
Condensing	3.2	1.6%	3.3	3.0%	8.8	39.8%
Solution of QPs	74.4	36.8%	74.5	68.5%	7.6	35.5%
Rest	1.76	0.9%	1.6	1.4%	2.3	10.5%

Table 2: A typical distribution of computing times. The absolute times given in seconds have been scaled to be independent of the number of iterations.

tables 1 and 2) with an advanced solver such as `DAESOL` (see Bauer 1999), but there is still a considerable difference to the formulation (22) with only one state and two control variables.

The solution of the QPs in the SQP scheme is also much more expensive for problem (17), as condensing does not reduce the number of variables actually given to the QP. If we do not perform condensing for problem (22), the computing time for "Solution of QPs" goes up to 68 seconds and almost reaches the level of problem (17).

To sum up, reformulation (22) is better suited for numerical calculations than (17), as it has a structure that can be better exploited by standard direct multiple shooting methods. Hence, in the following we will only use this formulation for our calculations.

We now solve the model to investigate the system dynamics dependent on the time-lag σ. In particular, we analyze the transition between instantaneous and delayed capital accumulation by solving (34) respectively (22) for different time-lags σ. Figure 1 shows optimized paths for time-lags σ ranging from 0 to 0.5. Consistent with the findings in section 2.2 the optimal paths converge monotonically towards the stationary state for $\sigma = 0$ and oscillatory and exponentially damped for $\sigma > 0$.

The continuous transition from monotonic to increasingly oscillatory optimal paths for increasing time-lags σ can be seen in figure 2. The exogenous parameters are identical to the calculations for figure 1. The interval for the time-lag $\sigma \in [0.1, 0.5]$ has been split into a grid of 400 equidistant points. For each of these σs the optimal control problem has been solved and the resulting graphs have been composed to the 3-dimensional plots

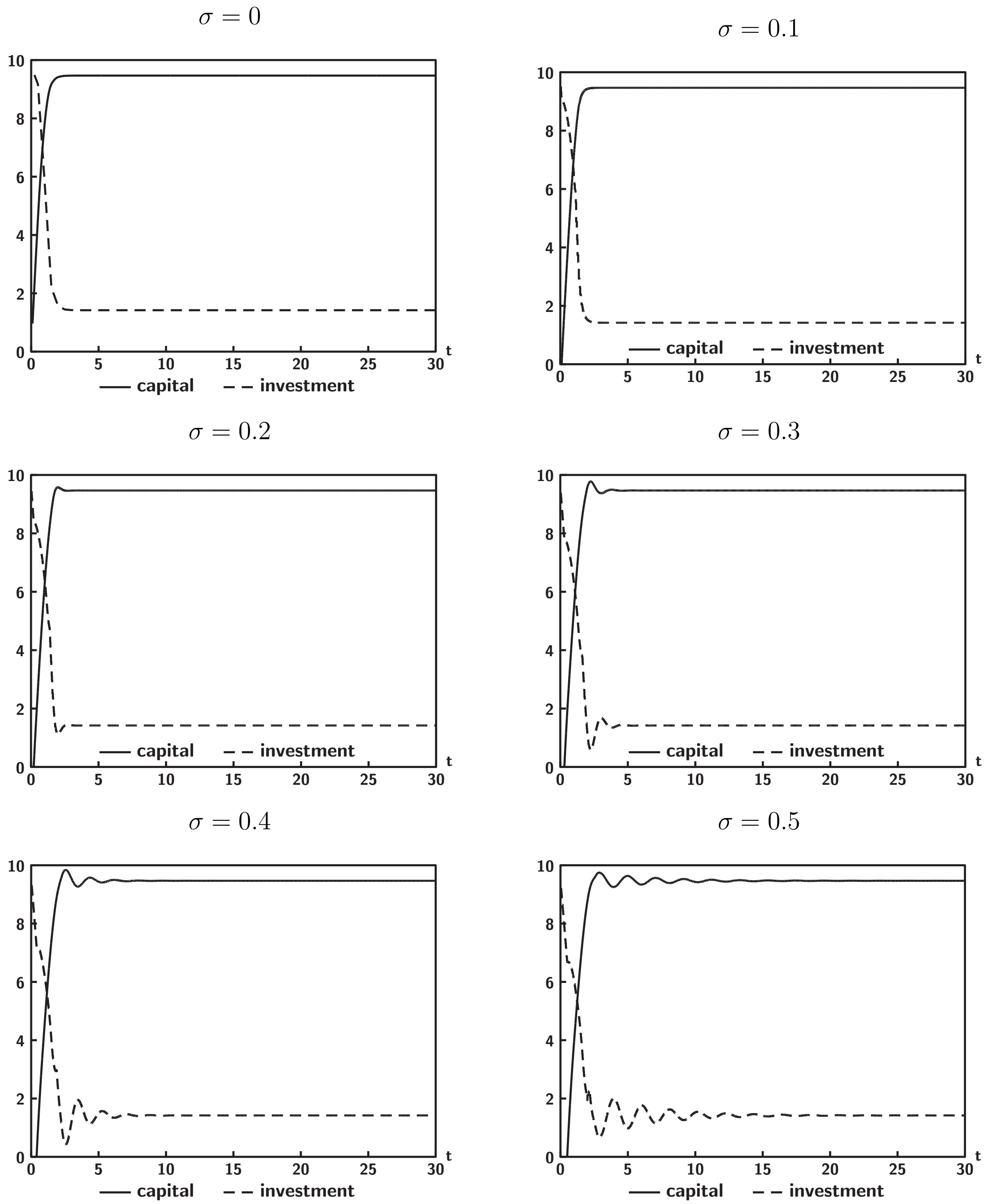

Figure 1: Optimal paths for capital and investment for selected time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation.

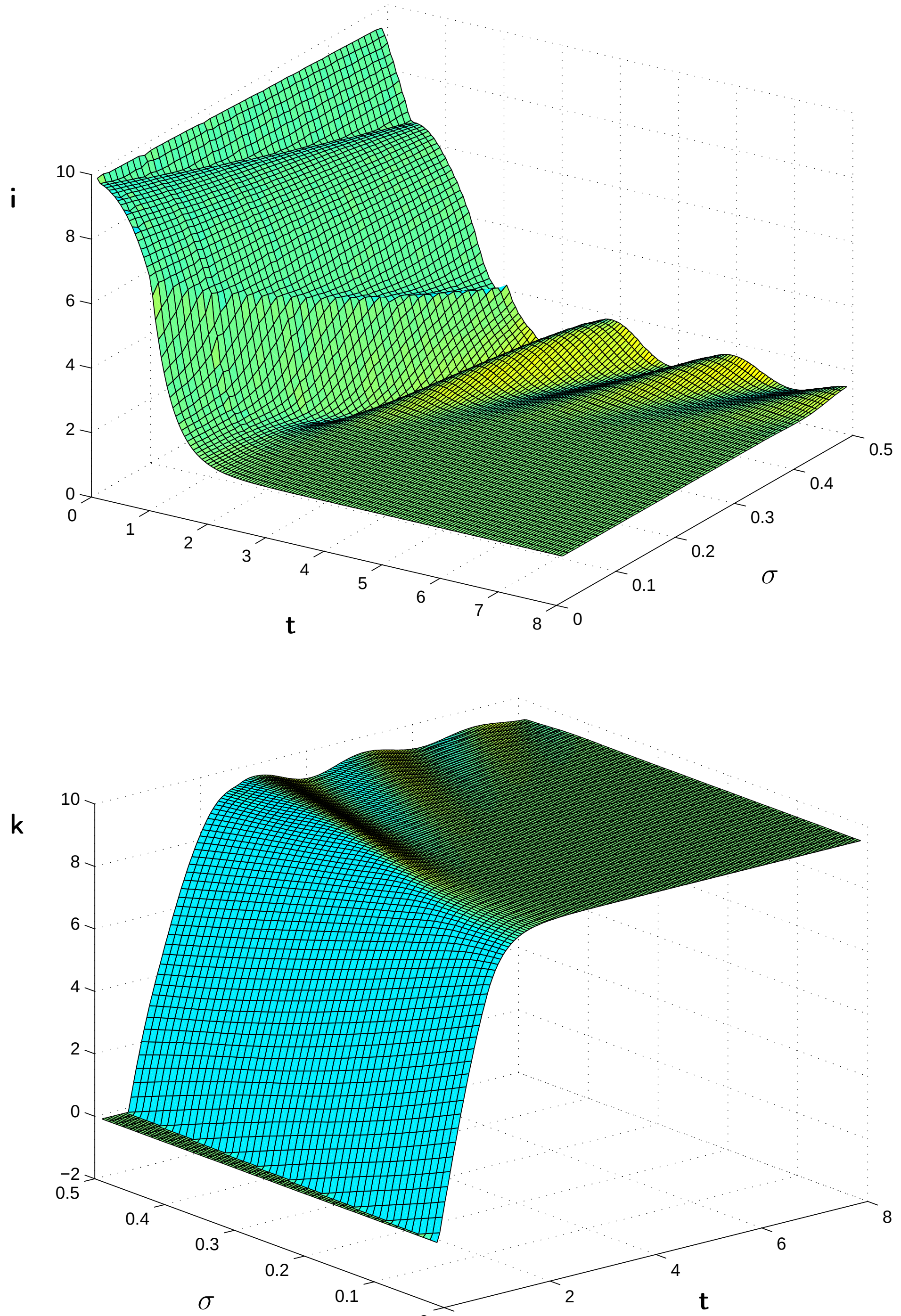

Figure 2: Optimal investment (top) and capital (bottom) paths for time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation. The third axis denotes increasing time-lags σ.

in figure 2. They show how the optimal paths evolve from monotonic to oscillatory paths for increasing time-lag σ.

4.2 The influence of the initial path on the optimal control of delayed pollution stock accumulation

The second model, first introduced in Winkler (2004), discusses the case of delayed pollution accumulation. The idea is that a joint output of production, which is released into the environment, accumulates there to a pollutant stock, which exhibits a negative effect on the economy. Although the following model has been inspired by the environmental problem of the emission of chlorofluorocarbons (CFCs), it is applicable to various stock pollutants. CFCs are a prime example of delayed accumulating stock pollutants. They have been widely used as cooling agents in refrigeration and air conditioning, as propellants in aerosols sprays and foamed plastics, and as solvents for organic matters and compounds. The CFCs have been valued because of their favorable chemical and biological characteristics. They are chemically inert, not inflammable and non-toxic. Unfortunately, in the stratosphere the CFCs cause the depletion of the ozone layer, which shields the earth's surface from ultraviolet radiation. Once released, the CFCs need 5–10 years to reach a height of about 30 km, where the depletion of the ozone layer starts. Hence, the stock of stratospheric CFCs reacts to the emissions of CFCs with a delay of 5–10 years.

Consider an economy with one non-producible input of production, for example, labor, which is given in a constant maximal amount $\bar{l}$ and distributed among two production processes in the economy. The first production process produces a consumption good c with constant returns to labor

$$c(t) = l_1(t) \ , \tag{35}$$

where l_1 denotes the amount of labor employed to the consumption good production. In addition, the production of each unit of consumption good gives rise to one unit of gross emissions e^{gross}:

$$e^{gross}(t) = c(t) = l_1(t) \ . \tag{36}$$

The second production process is an abatement process, which reduces net emissions e

$$e(t) = e^{gross}(t) - a(t) \ , \tag{37}$$

where a denotes the amount of emissions abated. Denoting the amount of labor employed to the abatement process by l_2, the amount of abated emissions is given by:

$$a(t) = \sqrt{\alpha l_2(t)} \ , \quad \alpha > 0 \ . \tag{38}$$

The net emissions e are considered to accumulate the pollution stock s with a time-lag σ. In addition, the pollution stock s decays at a constant rate γ

$$\dot{s}(t) = e(t-\sigma) - \gamma s(t) \ . \tag{39}$$

The stock of pollutant s exhibits a negative external effect on the economy, as it reduces the effective labor force l:

$$l(t) = \bar{l} - \beta s(t)^2 \ , \quad \beta > 0 \ . \tag{40}$$

In the case of CFCs, one might think of an increase in the rate of skin cancer with increasing stock of the pollutant, which prevents increasingly more people from working. Note that the pollution stock s exhibits increasing marginal damage. Given efficient production (i.e., the labor constraint holds with equality $l(t) = l_1(t)+l_2(t)$), consumption is given by

$$c(t) = c(e(t), s(t)) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2}\right] \ . \tag{41}$$

Again, we assume that the objective is to maximize intertemporal welfare, which is the discounted infinite integral of instantaneous welfare $V(c(t))$. Thus, the optimal control problem reads:

$$\max_{i(t)} \int_0^\infty V\left(\frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\left(\lambda - \beta s(t)^2 - e(t)\right) + \alpha^2}\right]\right) \exp[-\rho t] \ dt \tag{42a}$$

subject to

$$\dot{s}(t) = e(t-\sigma) - \gamma s(t) \ , \tag{42b}$$

$$e(t) = \xi(t) \ , \quad t \in [-\sigma, 0) \ , \tag{42c}$$

$$s(0) = s_0 \ . \tag{42d}$$

Again, the optimization problem (42) is of the form (1) and will be solved by the direct multiple shooting method. Here, the focus is on the dependence of the optimal paths on the initial path ξ. In particular, this is relevant in the context of pollution control, as the pollutant has in general already been emitted before pollution control becomes affective. Due to the additional moment of inertia of delayed control problems, the past emission path has to be taken into account. In the following we show the optimal emission paths for a numerical example of the optimization problem (42) for a constant, a linear, and a cyclical initial path. We choose $V = \ln c(t)$, $\bar{l} = 1$, $\alpha = 1$, $\beta = 0.005$, $\gamma = 0.1$, $\rho = 0.03$, $t_f = 200$, $s_0 = 10$, $\xi_{const} = 1.47459$, $\xi_{lin} = 1 + 0.0815485(t + 10)$ and $\xi_{cyc} = 1.39815 + \sin[0.9\pi(t + 10)]$. To be able to compare the results for these different initial paths, they have been chosen in such a way that the stock of pollution at time $t = \sigma = 10$ is identical for all three of them ($s(10) = s_\sigma = 13$).

Figure 3 shows the optimal paths of the pollution stock and the emissions in the case of delayed stock accumulation ($\sigma = 10$) for the three different initial paths ξ. The initial paths ξ are shown as the emission paths in the time interval $t \in [-10, 0]$ in figure 3. As already mentioned earlier, the path for the pollution stock in the time interval $t \in [0, 10]$ is completely determined by the initial value s_0, the initial path ξ and the equation of motion (42b). Hence, pollution control from time $t = 0$ on only affects the pollution stock after time $t = \sigma = 10$. This shows a fundamental feature of delayed optimal control

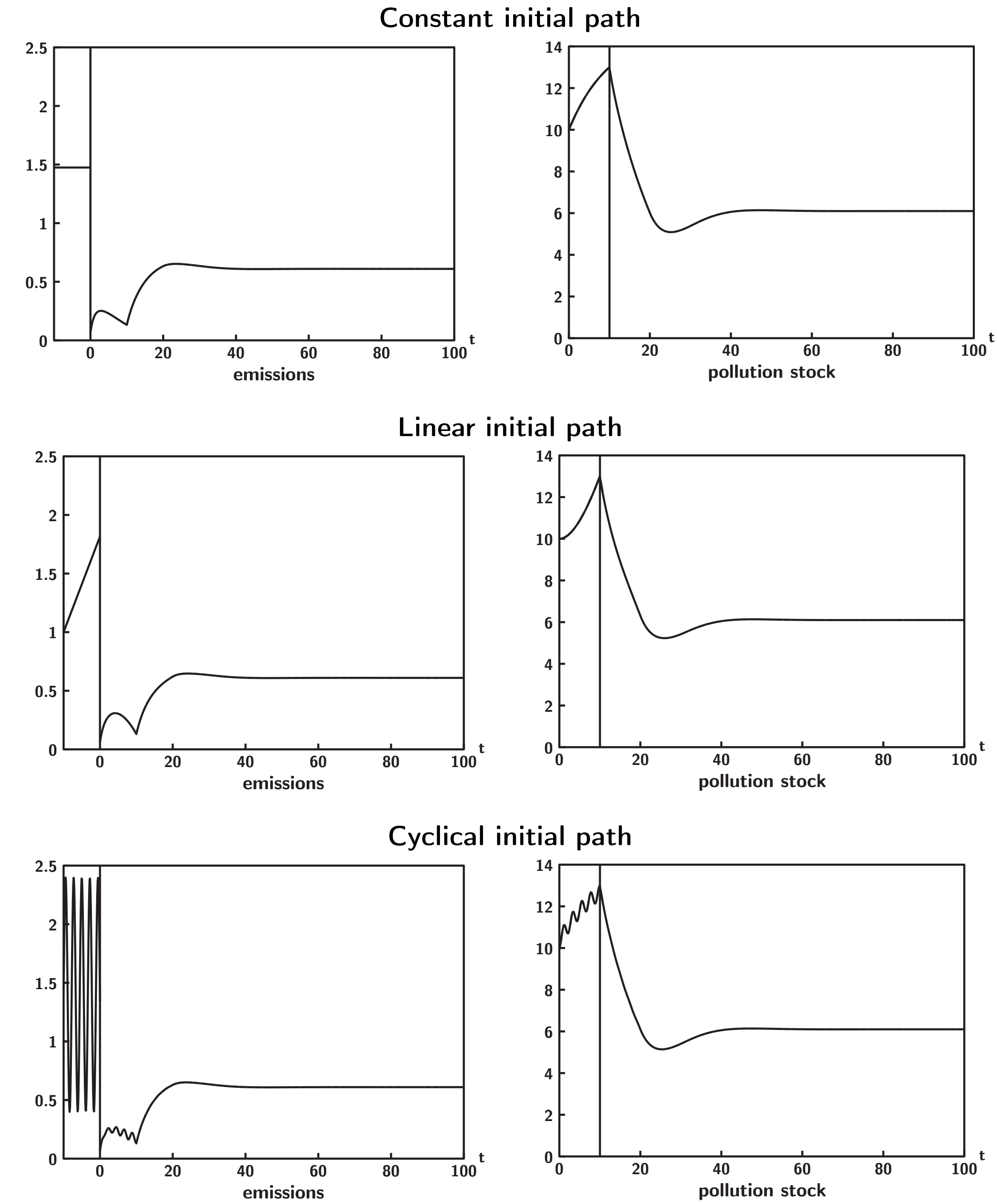

Figure 3: Optimal paths for the emissions (left) and the pollution stock (right) for constant (top), linear (middle) and cyclical (bottom) initial paths ξ.

problems: the system dynamics exhibits an additional moment of inertia as the stock reacts with a delay to the control.

In all three scenarios the pollution stock rises from their initial value $s_0 = 10$ to $s_\sigma = 13$ in the time interval $t \in [0, 10]$. Nevertheless, because of the different initial paths ξ, the path of the pollution stock is concave (ξ constant), convex (ξ linear) or oscillatory (ξ cyclical). Variations in the initial path ξ cause variations in the optimal system dynamics, although the pollution stock $s_\sigma = 13$ and the long-run stationary state remains unaltered. This is best seen in the case of a cyclical initial path, which induces corresponding oscillations in the optimal emission path (figure 3 bottom).

5 Conclusions

As known from the time-to-build literature, delayed optimal control problems with one stock and one control variable exhibit in general a qualitatively different system dynamics compared to instantaneous optimal control problems. While the optimal paths of the latter converge strictly monotonically towards the stationary state, the former exhibit oscillatory and exponentially damped optimal paths.

In this paper we have drawn attention to the numerical solution of delayed optimal control problems. Therefore, we have shown how delayed optimal control problems can be reformulated such that direct state-of-the-art methods can be applied. In particular, we presented two different problem reformulations and compared the performance of Bock's direct multiple shooting algorithm, implemented in the software package `MUSCOD-II`. While the first reformulation increases the dimensionality of the resulting optimization problem drastically by introducing as many new stock and control variables as the time horizon t_f, over which is optimized, is a multiple of the time-lag σ, the second reformulation only introduces one additional control variable, irrespective of the time horizon t_f and the time-lag σ. While the latter reformulation exhibits better computational performance, the former allows for intuitive explanations of some standard analytic results of the control-delayed optimal control problem.

Numerical optimization plays a crucial part in the analysis and understanding of delayed optimal control problems, as even the linear approximation of the system dynamics around the stationary state is not analytically tractable. As we understand the lack of application of delayed optimal control in economics to be (at least partly) a consequence of the analytical and numerical difficulties, we hope that this paper encourages broader research in this area. In fact, there a numerous applications in the field of economics alone. With two examples we have shown how to apply the method for the rigorous analysis of the transition from instantaneous to delayed capital accumulation and for the analysis of the influence of the initial path on the optimal time-lagged accumulation of a pollution stock. However, we also expect our numeric approach to be valuable for other fields of scientific endeavor.

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